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VII.

CONTRIBUTIONS FROM THE PHYSICAL LABORATORY OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

No. I.

FOCI OF LENSES PLACED OBLIQUELY.

By Prof. E. C. Pickering and Dr. Chas. H. Williams.

Presented, Feb. 9, 1875.

The following experiments were suggested by noticing that the spot of light of a reflecting galvanometer was thrown out of focus when the mirror turned slightly. The image in this case was formed by a lens near the mirror ; and, to obtain a distinct image, it was found that the ends of the scale must be brought much nearer the mirror, owing to the obliquity of the rays upon the lens. It was further noticed that the focus was greatly altered when the slit used was placed horizontally instead of vertically. To study the matter more carefully, the following apparatus was constructed.

To one end of a board about five feet long was fitted a small telescope; ten inches from this was placed a graduated circle, which rested horizontally on the board; and at its axis was fixed a screen of sheet iron, which stood vertically and had a hole in the centre, on a level with the telescope.

On one side of the screen, and covering the hole, was fixed a biconvex lens of 25.5 inches focus, so placed that, when the graduated circle was moved, the lens turned about a vertical axis which coincided with that of the circle. At the farther end of the board was placed a small gas flame, and between the lens and gas was a screen which moved over a scale giving the distance in inches from the axis of the lens. At the centre of this screen, on a level with the lens, telescope, and gas flame, and in the same straight line, two fine slits were cut, one vertical the other horizontal, intersecting each other in the middle; and in these slits filaments of silk were stretched lengthwise, to aid in focussing.

To use the instrument, the gas was lighted, then the screen with the cross-hairs was placed at the principal focus of the lens: that is, 25.5

inches from it. The graduated circle, carrying the lens, was now brought to zero, so that the rays from the cross should fall normally on the centre of the lens. Lastly, the telescope was focussed on the cross-hairs. The zero point of the graduated circle was obtained with great accuracy, by lighting the gas, then, placing the eye at some distance behind the gas flame, the graduated circle was moved till the reflections from the anterior and posterior surfaces of the lens exactly coincided with the flame ; in this way the true vertical as well as horizontal position was obtained. The instrument being thus adjusted, the graduated circle and lens were turned five degrees. The screen having the cross slits was now moved, by means of a rod attached to it, until the rays from the vertical slit were properly focussed by the observing telescope, the distance of the screen from the lens was read from the scale, the reading repeated three times, and the mean recorded. The same was afterward done for the rays from the horizontal slit. The graduated circle was then moved on, and the same readings repeated every five degrees. It was impossible to take readings from the vertical slit beyond 65° ; for, after that, the screen could not be brought near enough to the lens to focus the rays properly, and the image became quite indistinct; but with the horizontal slit the readings were continued to 85°. After completing this set of readings, the screen was placed at one and a half times its focal length from the lens, the graduated circle brought to zero, and the telescope focussed as before ; then the same readings were repeated every five degrees, also when the screen was at one half and at twice the focal distance.

Having obtained these readings, curves were constructed by the Graphical Method, the vertical distances being equal to the distance from screen to lens, and the horizontal to the angle through which the graduated circle was moved. As a test for the accuracy of the readings when the telescope was focussed for different points, all the readings were reduced, so as to be compared with those taken when the distance from lens to screen was equal to the focal length of the lens. This was done by means of the formula $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, in which u and v are the conjugate foci, and f the principal focus of the lens. In these experiments, $\frac{1}{u}$ is a constant, for, when the telescope has been once focussed, it remains fixed through that set of readings, and the reciprocal is easily found by $\frac{1}{u} = \frac{1}{f} - \frac{1}{v}$; that is, subtracting the reciprocal of the distance from screen to lens, when the angle is equal to zero, from the reciprocal of the principal focus of the lens. This reciprocal is to be added to those of the readings, and thus the readings of any set are rendered equivalent to those taken when the screen is at a distance from the lens equal to its principal focus. This being done, the greatest variation of any of the readings from the standard was found to be a little over one per cent.

The result of these measurements of the vertical slit are given in Table I., and of the horizontal slit in Table II. Column 1 gives the

TABLE I.

i.	.5*f.*	*f.*	1.5*f.*	2*f.*	*f.*	.5*f.*	1.5*f.*	2*f.*	Mean.
0°	12.7	25.5	38.2	51.0	25.5	25.5	25.5	25.5	25.50
5°	12.6	25.3	37.6	49.3	25.3	25.1	25.2	25.1	25.17
10°	12.3	24.3	35.6	46.5	24.3	23.9	24.3	24.3	24.20
15°	12.0	23.0	33.0	41.0	23.0	22.8	23.1	22.8	22.92
20°	11.5	21.9	30.1	37.5	21.9	21.1	21.6	21.6	21.55
25°	11.0	19.8	26.3	32.6	19.8	19.5	19.6	19.9	19.70
30°	10.2	17.7	23.0	27.0	17.7	17.1	17.7	17.7	17.55
35°	9.2	15.1	19.1	22.0	15.1	14.5	15.3	15.4	15.07
40°	8.1	12.8	15.5	17.0	12.8	11.9	12.9	12.8	12.60
45°	7.0	10.3	12.1	13.0	10.3	9.6	10.5	10.4	10.20
50°	5.7	8.1	9.2	9.5	8.1	7.3	8.2	8.0	7.90
55°	4.7	6.1	6.5	7.0	6.1	5.7	5.9	6.1	5.95
60°	3.7	4.3	4.4	4.7	4.3	4.3	4.1	4.3	4.25
65°	2.6	3.1	3.1	3.2	3.1	2.9	2.9	3.0	2.97
70°									
75°									
80°									
85°									

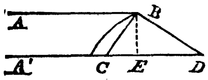
TABLE II.

i.	.5*f.*	*f.*	1.5*f.*	2*f.*	*f.*	.5*f.*	1.5*f.*	2*f.*	Mean.
0°	12.7	25.5	38.2	51.0	25.5	25.5	25.5	25.5	25.50
5°	12.8	25.4	38.1	50.5	25.4	25.9	25.5	25.4	25.55
10°	12.7	25.4	37.9	49.7	25.4	25.5	25.4	25.2	25.37
15°	12.7	25.0	37.1	48.9	25.0	25.5	25.0	25.0	25.12
20°	12.5	24.6	36.2	47.0	24.6	24.7	24.6	24.5	24.60
25°	12.4	24.0	35.1	44.7	24.0	24.3	24.1	23.8	24.05
30°	12.2	23.3	34.0	43.4	23.3	23.6	23.6	23.4	23.47
35°	12.0	22.5	32.2	41.4	22.5	22.8	22.7	22.8	22.70
40°	11.8	21.9	30.5	37.9	21.9	22.1	21.8	21.7	21.87
45°	11.5	20.9	28.9	34.7	20.9	21.1	21.0	20.7	20.92
50°	11.2	19.7	26.7	32.4	19.7	20.1	19.8	19.8	19.85
55°	10.9	18.4	25.2	30.0	18.4	19.1	19.0	18.9	18.85
60°	10.5	17.5	23.5	27.6	17.5	17.9	18.0	17.9	17.82
65°	10.1	16.6	21.5	24.7	16.6	16.8	16.8	16.7	16.72
70°	9.8	15.7	20.0	22.7	15.7	16.0	15.9	15.7	15.82
75°	9.5	14.8	18.2	20.5	14.8	15.2	14.7	14.6	14.82
80°	9.0	13.7	16.8	18.7	13.7	14.0	13.8	13.7	13.80
85°	8.6	12.8	15.3	17.0	12.8	13.0	12.8	12.8	12.85

angle of incidence, the next four columns the observed conjugate focus, u, or position of the slit when the telescope was focussed on a point seen through the lens at a distance of $.5\,f$, f, $1.5\,f$, and $2\,f$, in turn. The next four columns give the computed value of f, assuming that a lens placed obliquely conforms to the law $\frac{1}{u}+\frac{1}{v}=\frac{1}{f}$, as well as when in the ordinary position. The result justifies this assumption; for the four values of f are nearly coincident, and agree well with the mean given in the last column. The phenomena are thus greatly simplified, since we have now only to consider the case of the principal focal distances, or that the incident ray forms a parallel beam.

To represent these results theoretically, let us suppose the slits and lens so small, compared with their distance apart, that we may neglect all aberration except that due to the obliquity of the incidence. Considering first the case of the vertical slit, let Fig. 1 represent the section of a horizontal plane passing through the centre of the lens. Then let D represent the position of the slit when the emergent rays are parallel; that is, when AB is parallel to $A'C$. Now $CD=f'$ is the new focal length which is to be determined. Call f the principal focal distance, n the index of refraction, i and r the angles of incidence and refraction of the light on entering the lens, and r' and i' the corresponding angles on its emergence. Call also A the angle between the two surfaces of the lens at its edge, or of the two surfaces where pierced by the ray. Then, by the law of refraction, $\sin i = n \sin r$, and $\sin i' = n \sin r' = n \sin (r + A) = n \sin r + n\,A \cos r = \sin i + n\,A \cos r$, since $r' = A + r$ and $\sin A$ being very small may be regarded as equal to A. Again, $\sin i' - \sin i = \cos i\,(i' - i) = n\,A \cos r$, and hence, $\frac{i'-i}{A} = \frac{-n \cos r}{\cos i}$.

Fig. 1.



Now, in the triangle BCD we have $BDC = i - i' - A$, $BCD = 90 - i$, and BD sensibly equal to f'. Again, $BC = fA\,(n-1)$, for by the formula for lenses $\frac{1}{f} = (n-1)\left(\frac{1}{R}+\frac{1}{R'}\right)$, or $f = \frac{R}{2\,(n-1)}$, but $A = \frac{2\,BC}{R}$ $\therefore f = \frac{BC}{A\,(n-1)}$, or $BC = fA\,(n-1)$. Since the sides are proportional to the sines of the opposite angles $BD : BC = \sin BCD : \sin BDC$, or $f' : fA\,(n-1) = \sin (90-i) : i-i'-A$ or $f' = f.\frac{A\,(n-1) \cos i}{i-i'-A}$; dividing by A, and substituting the value of $\frac{i-i'}{A}$ given above, we have $f' = f\,\frac{(n-1) \cos^2 i}{n \cos r - \cos i} = f \cdot \frac{(n-1) \cos^2 i}{\sqrt{n^2 - \sin^2 i} - \cos i} = f \cdot$

$$\frac{(n-1)\ \cos^2 i}{n^2-1}\ (\sqrt{n^2-\sin^2 i}+\cos i)=f\ \frac{\cos^2 i}{n-1}\ (\sqrt{n^2-\sin^2 i}+\cos i).$$

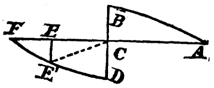
By means of this formula, the value of f' was computed for every 5° for $n = 1.5$; and the results, calling $f = 100$, are given in column 2 of Table III. Column 1 of the same table gives the corresponding angle of incidence, and column 3 the rate of change of f for small changes of n, or $\frac{df}{dn}$. This is only serviceable if we wish to compute the foci of lenses of various indices, but it is applicable only for value of n near 1.5 or 1.6. As an example, suppose we wish the focus of a lens having an index of refraction = 1.57, and inclined 45°. Then $f' = 40.6 + .07 \times .6 = 41.0$. The values in column 4 are computed in this way, and give the foci for the lens actually employed, whose index was assumed to be equal to 1.55. To compare these results with observation, the last column of Table I. was reduced by dividing by 25.5, the principal focal distance. The differences or errors are given in column 6, which show the close agreement with theory. From these it appears that the deviations are probably mainly due to accidental errors, the preponderance of negative values rendering it probable that the focal distance 25.5 was taken as too small.

TABLE III.

i.	1.5.	$\frac{df}{dn}$	1.55.	Obs.	ε.
0°	100.0	0.0	100.0	100.0	0.0
5°	98.9	0.0	98.9	98.7	—0.2
10°	96.1	0.1	96.1	94.9	—1.2
15°	91.2	0.2	91.3	89.8	—1 5
20°	84.8	0.2	84.9	84.5	—0.4
25°	77.0	0.3	77.1	77.2	+0.1
30°	68.4	0.4	68.6	68.8	+0.2
35°	59.2	0.5	59.4	59.1	—0.3
40°	49.8	0.6	50.1	50.3	+0.2
45°	40.6	0.6	40.9	40.0	—0.9
50°	32.0	0.6	32.3	31.0	—1.3
55°	24.1	0.6	24.4	23.3	—1.1
60°	17.1	0.5	17.3	16 7	—0.6
65°	11.3	0.4	11.5	11.6	+0.1
70°	7.0	0.3	7.1		
75°	3.8	0.2	4.0		
80°	1.6	0.1	1.6		
85°	0.4	0.0	.4		
90°	0.0	0.0	0.0		

The case of the horizontal slit is more complicated, since the rays no longer remain in one plane. Considering only those rays in the vertical plane passing through the axis around which the lens turns, and one point of the slit, we see that they will strike the lens at an angle of incidence about equal to i, will traverse it in a plane which we will call the plane of refraction inclined to the first plane $i - r$, and finally emerge in a plane parallel to the first. The plane of refraction will intersect the lens along two circles whose distance apart at the centre will be greater than the thickness of the lens in the ratio of cos r to 1; hence their radii R' will be less than the radius of curvature R of the surfaces of the glass in the same ratio, or $R' = R \cos r$. Again, the apparent index of refraction n' will be different, and since $\frac{1}{f} = \frac{2(n-1)}{R}$ and $\frac{1}{f'} = \frac{2(n'-1)}{R'}$, we have $f' = f\frac{n-1}{n'-1} \cdot \frac{R'}{R} = f \cos r \frac{n-1}{n'-1}$. It therefore only remains to determine n', the apparent index of refraction. As the problem is one in spherical trigonometry, suppose a sphere described around the centre of the lens and projected in Fig. 2, the eye lying in the axis of the lens prolonged. Let $CA = i$, the angle through which the lens has been turned, and $CE = r$, the corresponding angle of refraction. Then if the surface of the glass is vertical, as at the centre of the lens, the incident ray will be AC and the refracted ray CE. Next suppose the surface slightly inclined by the amount $CD = BC = v$, as is the case for the upper and lower parts of the lens. $AB = i'$ will now be the angle of incidence; and, to construct the refracted ray, we have first the condition that $\sin i' = n \sin r'$, and secondly that the incident and refracted ray shall lie in the same plane with the normal BCD. To construct it, pass a plane through the normal BC and the incident ray AC, which will intersect the sphere along the great circle AB and FD; on this, lay off $DE' = r'$ such that $n \sin r' = \sin i'$, but, as v is infinitesimal, i' will be sensibly equal to i, and r' to r. Now in the right-angled spherical triangle FCD, $\sin CD = \sin DF \sin CFD$, or $\sin v = v = \sin i \times F$, or $F = \frac{v}{\sin i}$; and in the triangle FEE', $\sin EE' = \sin FE' \sin EFE'$, or $EE' = \sin (i - r) F$, or, substituting the value of F just found, $EE' = v \frac{\sin (i-r)}{\sin i}$. Calling i'' and r'' the angles of incidence and refraction of the ray with regard to the section of the lens made by the plane of refraction, then i'' will not equal BC, but will be the angle which, when projected on the plane of the section of the lens, will be BC or $i'' \cos r = BC \therefore i'' = \frac{v}{\cos r}$. Again EE' is the angle

Fig. 2.



through which the ray is bent, or $i'' - r'' = v \frac{\sin(i-r)}{\sin i}$, or subtracting, $r'' = v \left[\frac{1}{\cos r} - \frac{\sin(i-r)}{\sin i}\right] = v \frac{\sin i - \sin(i-r) \cos r}{\sin i \cos r}$; but $n' = \frac{i''}{r''}$, which, with the above values, gives $n' = \frac{\sin i}{\sin i - \cos r \sin(i-r)}$. Substituting this value in the equation $f' = f \cos r \frac{(n-1)}{(n'-1)}$ gives $f' = f (n'-1) \frac{\sin i - \sin(i-r) \cos r}{\sin(i-r)}$. This would be the focal distance if the rays on emerging remained in the plane of the section of the lens. But they pass into a plane inclined to this $i - r$, hence the observed focus f'' will be such that when projected on the plane of the section it will equal f', or $f'' \cos(i-r) = f'$. Hence finally $f'' = f (n-1) \frac{\sin i - \sin(i-r) \cos r}{\sin(i-r) \cos(i-r)}$. This last step may be open to criticism, but the close agreement with observation seems to justify it.

In Table IV., this formula is compared with observation, as the law for the vertical slit is compared in Table III. The columns in the two tables correspond, and it will be noticed that the agreement is very close.

TABLE IV.

i.	1.5.	$\frac{df}{dn}$	1.55.	Obs.	ε.
0°	100.0	0 0	100.0	100.0	0.0
5°	99.9	0.0	99.9	100.2	+0.3
10°	99.2	0.2	99.3	99.5	+0.2
15°	97.7	0.3	97.8	98.5	+0.7
20°	96.1	0.5	96.3	96.4	+0.1
25°	93.7	0.6	94.2	94.3	+0.1
30°	91.1	0.8	91.5	92.0	+0.5
35°	88.3	1.0	88.8	89.0	+0.2
40°	84.7	1.2	85.3	85.7	+0.4
45°	81.1	1.5	81.8	82.0	+0.2
50°	77.2	1.7	78.0	77.8	—0.2
55°	73.2	1.9	74.1	73 9	—0.2
60°	69.0	2.1	70.0	69.8	—0.2
65°	64.7	2.4	65.9	65.5	—0.4
70°	60.4	2.6	61.7	62.0	+0.3
75°	56.3	2.8	57.7	58.1	+0.4
80°	52.1	3.0	53.6	54.1	+0.5
85°	48.3	3.2	49.9	50.4	+0 5
90°	44.8	3.4	46.5		

The principal practical application of these results is to photographic lenses. It will be seen that a single lens, even if perfectly corrected for spherical and chromatic aberration, is still subject to this defect. Constructing the curves with polar co-ordinates, taking the radius vector equal to the focal length and its angle equal to the angle of incidence, we obtain a line every point of which would be in focus at the same time. This shows that in a photographic camera for lines passing through the axis, corresponding to the vertical slit, the surface instead of being a plane should have a radius of curvature of only .3 the focus. For lines perpendicular to these, or circles concentric with the centre, corresponding to the horizontal slit, the curvature should be .7 the focus. We also see the importance of having telescope lenses carefully centred, and why the images of stars, if this adjustment is not exact, are elliptical instead of circular.

Since writing the above, a further application of these formulas has been suggested in the case of the eye, that the imperfect vision at a distance from the centre of vision may be due to the rays passing obliquely through the lens. It will also be noticed that the curvature of the retina corresponds nearly with that which would give the best vision. As stated above, for radial lines the radius of curvature should be about .3, and for concentric circles .7, its distance from the lens. The actual curvature in the normal eye is about .5, or the mean of these values.